Activity 12
Complex Numbers 3
Conjugate, modulus and argument
Dr. Juan Manuel Campos Sandoval

Name____

Theorem **Properties of Complex Conjugate**

Let z be a complex number. Then

(1) z is real if and only if $\bar{z} = z$.

(2) $\bar{\bar{z}} = z$ (3) $z\bar{z} = |z|^2$

(4) $|\bar{z}| = |z|$ (5) $\arg \bar{z} = -\arg z \quad (z \neq 0)$

(6) $z + \bar{z} = 2\,\mathrm{Re}(z)$ (7) $z - \bar{z} = 2i\,\mathrm{Im}(z)$

Theorem **Properties of Complex Conjugate (Continued)**

Let z_1 and z_2 be two complex numbers. Then

(8) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(9) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(10) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}} \qquad (z_2 \neq 0)$

Theorem **Properties of Modulus**

Let z be a complex number. Then

a) $|\bar{z}| = |z|$ b) $z\bar{z} = |z|^2$

c) $|z_1 z_2| = |z_1||z_2|$ d) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, z_2 \neq 0.$

e) $|z^n| = |z|^n$

1. Find the modulus of $e^{j\theta}$

2. Show that $(e^{j\theta})^{-1} = e^{-j\theta}$ (without using exponential properties)

3. Show that $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$ $\qquad$ $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

4. Write down the complex conjugates of:

i) $5 + 3j$ $\qquad$ ii) $\dfrac{1}{3 - 4j}$ $\qquad$ iii) $\dfrac{j}{j + 2}$ $\qquad$ iv) $\dfrac{j - 2}{3 + 2j}$

v) $\left(\dfrac{j - 2}{3 + 2j}\right)^4$ $\qquad$ vi) $\dfrac{1}{j(2 - 5j)^2}$

5. Put into a + jb form

i) $(2 + 5j)(4 - 3j)$ $\qquad$ ii) $(4 - j)(1 + j)(3 + 4j)$ $\qquad$ iii) $\dfrac{1}{3 - 4j}$

iv) $\dfrac{2 - 5j}{1 + 4j}$ $\qquad$ iv) $\dfrac{-1 + 3j}{(3 - 2j)(2 + j)}$

6. Evaluate

i) $\dfrac{1}{4 - 3j} + \dfrac{1}{4 + 3j}$ $\qquad$ ii) $\dfrac{2 + j}{4 - 3j} - \dfrac{2 - j}{4 + 3j}$ $\qquad$ iii) $\dfrac{1}{(5 + 3j)(5 - 3j)}$

and explain why each is either purely real or purely imaginary.

7. Simplify the complex number $\dfrac{2 - j}{3 + j} + \dfrac{1 + j}{1 - j}$. Find the modulus and argument of the result.

8. If $|z_1| = 5$, Arg $z_1 = \pi/3$, $|z_2| = 3$, Arg $z_2 = \pi/4$, find the Cartesian forms of z_1 and z_2 and the values of:

i) $|z_1 z_2|$ $\qquad$ ii) $\left|\dfrac{z_1}{z_2}\right|$ $\qquad$ iii) $|z_1^2|$

iv) Arg $(z_1 z_2)$ $\qquad$ v) Arg $\left(\dfrac{z_1}{z_2}\right)$ $\qquad$ vi) Arg $\bar{z}_1$

9. Show that multiplication by j rotates a complex number through $\frac{\pi}{2}$ in the anticlockwise direction and division by j rotates it $\frac{\pi}{2}$ in the clockwise direction.

10. If $z_1 = 3\angle\left(\frac{\pi}{6}\right)$, $z_2 = 2\angle\left(\frac{\pi}{18}\right)$, $z_3 = 1\angle\left(\frac{\pi}{3}\right)$

Find the polar form and the modulus of

i) $z_1 z_2$ ii) $\frac{z_1}{z_2}$ iii) $z_1 z_2 z_3$

iv) $\frac{z_2^2}{z_3}$

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Theorem **Properties of Complex Conjugate**

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Theorem **Properties of Modulus**

Let z be a complex number. Then

a) $|\bar{z}| = |z|$ b) $z\bar{z} = |z|^2$

c) $|z_1 z_2| = |z_1||z_2|$ d) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, z_2 \neq 0.$

e) $|z^n| = |z|^n$

1. Find the modulus of $e^{j\theta}$

Solution

$$|e^{j\theta}| = |\cos\theta + j\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = \mathbf{1}$$

2. Show that $(e^{j\theta})^{-1} = e^{-j\theta}$ (without using exponential properties)

Solution

$$(e^{j\theta})^{-1} = (\cos\theta + j\sin\theta)^{-1} = \frac{1}{\cos\theta + j\sin\theta}$$

$$= \frac{\cos\theta - j\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - j\sin\theta = \cos(-\theta) + j\sin(-\theta) = \mathbf{e^{-j\theta}}$$

3. Show that $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$ $\qquad$ $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

Solution

We have

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Adding these identities gives

$$2\cos\theta = e^{j\theta} + e^{-j\theta} \text{ so } \boldsymbol{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Subtracting the results gives

$$2j\sin\theta = e^{j\theta} - e^{-j\theta} \text{ so } \boldsymbol{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

4. Write down the complex conjugates of:

i) $5 + 3j$ $\qquad$ ii) $\dfrac{1}{3 - 4j}$ $\qquad$ iii) $\dfrac{j}{j + 2}$ $\qquad$ iv) $\dfrac{j - 2}{3 + 2j}$

v) $\left(\dfrac{j - 2}{3 + 2j}\right)^4$ $\qquad$ vi) $\dfrac{1}{j(2 - 5j)^2}$

Where appropriate put both the original complex number and its conjugate into a + jb form and check your results.

Note that you are **not** asked to write the complex conjugates in the form a + jb, but simply to state the complex conjugate of each given number. For this you only have to **change the sign of j** wherever it occurs. **Then** put the results in a + jb form and check against the original number in a + jb form.

Solution

i) If $z = 5 + 3j$ then we get immediately $\bar{z} = 5 + 3(-j) = 5 - 3j$

ii) If $z = \dfrac{1}{3 - 4j}$ then $\bar{z} = \dfrac{1}{3 - 4(-j)} = \mathbf{\dfrac{1}{3 + 4j}}$

Now

$$\frac{1}{3 - 4j} = \frac{3 + 4j}{9 + 16} = \frac{3}{25} + \frac{4}{25}j$$

while

$$\frac{1}{1 + 3j} = \frac{3 - 4j}{9 + 16} = \frac{3}{25} - \frac{4}{25}j$$

which is just the complex conjugate of $\dfrac{3}{25} + \dfrac{4}{25}j$ as required.

iii) If $z = \dfrac{j}{j + 2}$ then $\bar{z} = \dfrac{-j}{-j + 2} = \mathbf{\dfrac{j}{j - 2}}$

Now

$$\frac{j}{j + 2} = \frac{j(2 - j)}{4 + 1} = \frac{1}{5} + \frac{2}{5}j$$

while

$\dfrac{j}{j - 2} = \dfrac{j(-2 - j)}{1 + 4} = \dfrac{1}{5} - \dfrac{2}{5}j$ the complex conjugate of $\dfrac{1}{5} + \dfrac{2}{5}j$ as required.

iv) If $z = \dfrac{j - 2}{3 + 2j}$ then $\bar{z} = \dfrac{2 + j}{-3 + 2j} = \dfrac{(2 + j)(-3 - 2j)}{9 + 4} = \dfrac{-6 - 4j - 3j + 2}{13}$

$$= \mathbf{-\frac{4}{13} - \frac{7}{13}j}$$

You can confirm that $z = \dfrac{j - 2}{3 + 2j} = -\dfrac{4}{13} + \dfrac{7}{13}j$ and so obtain $\bar{z} = -\dfrac{4}{13} - \dfrac{7}{13}j$ directly as a check.

v) With $z = \left(\frac{j-2}{3+2j}\right)^4$ we have $\bar{z} = \left(\frac{-j-2}{3-2j}\right)^4 = \left(\mathbf{\frac{4}{13} + \frac{7}{13}j}\right)^4$

We know from iv) that $\frac{j-2}{3+2j} = -\frac{4}{13} + \frac{7}{13}j$ and so

$$\left(\frac{j-2}{3+2j}\right)^4 = \left(-\frac{4}{13} + \frac{7}{13}j\right)^4$$

and so it is obvious directly that

$$\bar{z} = \left(-\frac{4}{13} - \frac{7}{13}j\right)^4 = \left(\frac{4}{13} + \frac{7}{13}j\right)^4$$

without having to expand to check.

vi) With $z = \frac{1}{j(2-5j)^2}$ we have $\bar{z} = \mathbf{\frac{1}{-j(2+5j)^2}}$

Now

$$\frac{1}{j(2-5j)^2} = \frac{1}{20-21j} = \frac{20}{841} + \frac{21}{841}j$$

While

$$\frac{1}{-j(2+5j)^2} = \frac{1}{20+21j} = \frac{20}{841} - \frac{21}{841}j$$

as required. Note that in fact we could have worked entirely with the reciprocals in this case, without explicitly converting to a + jb form.

5. Put into a + jb form

i) $(2+5j)(4-3j)$ ii) $(4-j)(1+j)(3+4j)$ iii) $\frac{1}{3-4j}$

iv) $\frac{2-5j}{1+4j}$ iv) $\frac{-1+3j}{(3-2j)(2+j)}$

Solution

i) $(2+5j)(4-3j) = 8 - 6j + 2j - 15j^2 = 8 + 14j + 15 = \mathbf{23 + 14j}$

ii) $(4 - j)(1 + j)(3 + 4j) = (4 - j)(3 + 7j - 4) = (4 - j)(-1 + 7j)$

$$= -4 + 28j + j + 7 = \mathbf{3 + 29j}$$

Note how higher degree products can always be reduced to quadratic multiplication in complex numbers.

iii) $\dfrac{1}{3 - 4j} = \dfrac{1}{3 - 4j}\dfrac{3 + 4j}{3 + 4j} = \dfrac{3 + 4j}{9 + 16} = \mathbf{\dfrac{3}{25} + \dfrac{4}{25}\,j}$

iv) $\dfrac{2 - 5j}{1 + 4j} = \dfrac{2 - 5j}{1 + 4j}\dfrac{1 - 4j}{1 - 4j} = \dfrac{(2 - 5j)(1 - 4j)}{1 + 16} = \dfrac{-18 - 13j}{17} = \mathbf{-\dfrac{18}{17} - \dfrac{13}{17}\,j}$

iv) In this case it helps to work out the denominator first

$$\frac{-1 + 3j}{(3 - 2j)(2 + j)} = \frac{-1 + 3j}{8 - j} = \frac{(-1 + 3j)(8 + j)}{64 + 1}$$

$$= \frac{-8 - j + 24j - 3}{65} = \frac{-11 + 23j}{65} = \mathbf{-\frac{11}{65} + \frac{23}{65}\,j}$$

6. Evaluate

i) $\dfrac{1}{4 - 3j} + \dfrac{1}{4 + 3j}$ ii) $\dfrac{2 + j}{4 - 3j} - \dfrac{2 - j}{4 + 3j}$ iii) $\dfrac{1}{(5 + 3j)(5 - 3j)}$

and explain why each is either purely real or purely imaginary.

Solution

i) $z = \dfrac{1}{4 - 3j} + \dfrac{1}{4 + 3j} = \dfrac{4 + 3j + 4 - 3j}{(4 - 3j)(4 + 3j)}$ on taking the common denominator

$$= \frac{8}{16 + 9} = \mathbf{\frac{8}{25}}$$

This is purely real, as we would expect from the original form which is clearly its own complex conjugate, $z = \bar{z}$. In fact z is of the form $z = w + \bar{w} = 2\,\mathrm{Re}\,(w)$.

ii) $z = \dfrac{2 + j}{4 - 3j} - \dfrac{2 - j}{4 + 3j} = \dfrac{(2 + j)(4 + 3j) - (2 - j)(4 - 3j)}{(4 - 3j)(4 + 3j)}$

$$= \frac{8 + 10j - 3 - (8 - 10j - 3)}{25} = \frac{20j}{25} = \mathbf{\frac{4}{5}\,j}$$

From the original form we see that $z = -\bar{z}$ and so $\dfrac{2 + j}{4 - 3j} - \dfrac{2 - j}{4 + 3j}$ is purely imaginary, as we have found from the calculation. In fact z is of the form $z = w - \bar{w} = 2\,\mathrm{Im}\,(w)\,j$.

iii) $z = \dfrac{1}{(5 + 3j)(5 - 3j)} = \dfrac{1}{25 + 9} = \mathbf{\dfrac{1}{34}}$ which confirms that z is real as is clear from the original form. In fact z is of the form $\dfrac{1}{w\bar{w}} = \dfrac{1}{|w|^2}$ and this is real and positive.

7. Simplify the complex number $\dfrac{2 - j}{3 + j} + \dfrac{1 + j}{1 - j}$. Find the modulus and argument of the result.

Solution

$$z = \frac{2 - j}{3 + j} + \frac{1 + j}{1 - j} = \frac{(2 - j)(1 - j) + (1 + j)(3 + j)}{(3 + j)(1 - j)}$$

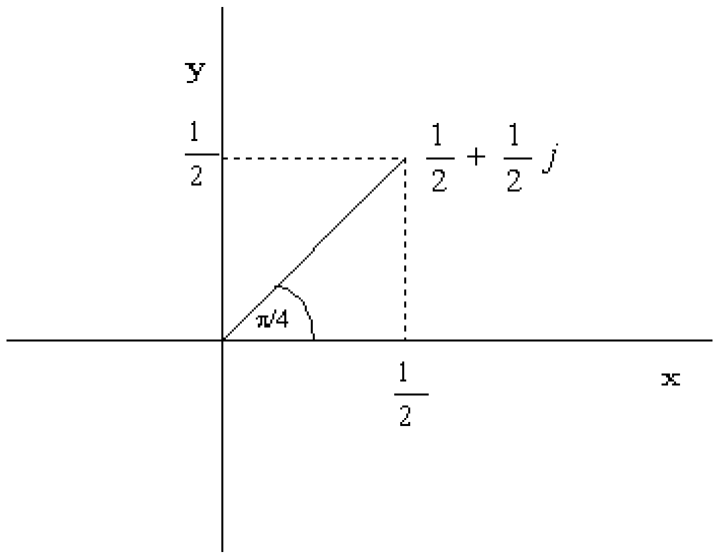
$$= \frac{2 - 3j - 1 + 3 + 4j - 1}{3 - 2j + 1} = \frac{3 + j}{4 - 2j} = \frac{3 + j}{4 - 2j}\,\frac{4 + 2j}{4 + 2j} = \frac{12 - 2 + 10j}{16 + 4}$$

$$= \frac{10 + 10j}{20} = \mathbf{\frac{1}{2} + \frac{1}{2}\,j}$$

For the modulus we have

$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \mathbf{\frac{1}{\sqrt{2}}}$$

The argument is seen to be $\frac{\pi}{4}$ from the figure.



8. If $|z_1| = 5$, Arg $z_1 = \pi/3$, $|z_2| = 3$, Arg $z_2 = \pi/4$, find the Cartesian forms of z_1 and z_2 and the values of:

i) $|z_1 z_2|$ ii) $\left|\frac{z_1}{z_2}\right|$ iii) $|z_1^2|$

iv) Arg $(z_1 z_2)$ v) Arg $\left(\frac{z_1}{z_2}\right)$ vi) Arg $\bar{z}_1$

Solution

Remember, for multiplication in polar form:

Multiply moduli
Add arguments

For division in polar form:

Divide moduli
Subtract arguments

Given $|z_1|$ and Arg z_1 we can write z_1 as $z_1 = |z_1| \angle(\text{Arg } z_1)$.

So

$$z_1 = 5\angle\left(\frac{\pi}{3}\right) = 5\left(\cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right)\right) = \mathbf{5\left(\frac{1}{2} + j\,\frac{\sqrt{3}}{2}\right)}$$

Similarly

$$z_2 = 3\angle\left(\frac{\pi}{4}\right) = 3\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right) = \mathbf{3\left(\frac{1}{\sqrt{2}} + j\,\frac{1}{\sqrt{2}}\right)}$$

i) $|z_1 z_2| = |z_1|\,|z_2| = 5 \times 3 = \mathbf{15}$

ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \mathbf{\frac{5}{3}}$

iii) $|z_1^2| = |z_1|^2 = 5^2 = \mathbf{25}$

iv) $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{\pi}{3} + \frac{\pi}{4} = \mathbf{\frac{7\pi}{12}}$

v) $\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) = \frac{\pi}{3} - \frac{\pi}{4} = \mathbf{\frac{\pi}{12}}$

vi) $\operatorname{Arg} \bar{z}_1 = -\operatorname{Arg} z_1 = \mathbf{-\frac{\pi}{3}}$

9. Show that multiplication by j rotates a complex number through $\frac{\pi}{2}$ in the anticlockwise direction and division by j rotates it $\frac{\pi}{2}$ in the clockwise direction.

Solution

Let $z = r\angle(\theta)$ be any complex number, with modulus r and argument θ (not necessarily the principal value). Since $j = 1\angle\left(\frac{\pi}{2}\right)$ we have

$$jz = 1\angle\left(\frac{\pi}{2}\right) r\angle(\theta) = r\angle\left(\theta + \frac{\pi}{2}\right)$$

which shows that the argument of jz is $\theta + \frac{\pi}{2}$ - ie **rotated anti-clockwise through $\frac{\pi}{2}$** .

On the other hand

$$\frac{z}{j} = \frac{r\angle(\theta)}{1\angle\left(\frac{\pi}{2}\right)} = r\angle\left(\theta - \frac{\pi}{2}\right)$$

So the argument of $\frac{z}{j}$ is $\theta - \frac{\pi}{2}$, ie **rotated clockwise through $\frac{\pi}{2}$** .

10. If $z_1 = 3\angle\left(\frac{\pi}{6}\right)$, $z_2 = 2\angle\left(\frac{\pi}{18}\right)$, $z_3 = \angle\left(\frac{\pi}{3}\right)$

Find the polar form and the modulus of

i) $z_1 z_2$ ii) $\frac{z_1}{z_2}$ iii) $z_1 z_2 z_3$

iv) $\frac{z_2}{z_3^2}$ v) $\frac{z_2 z_3}{z_1}$ vi) $\frac{z_1^2 z_3}{z_2}$

Solution

We have to take all the angles in consistent form, and perhaps degrees are most convenient, as we don't have to deal with fractions then. So write: $z_1 = 3\angle(30\pi)$, $z_2 = 2\angle(10\pi)$, $z_3 = \angle(60\pi)$. Then

i) $z_1 z_2 = 3\angle(30\pi)\ 2\angle(10\pi) = 6\angle(30\pi + 10\pi) = \mathbf{6\angle(40\pi)}$

ii) $\frac{z_1}{z_2} = \frac{3\angle(30\pi)}{2\angle(10\pi)} = \frac{3}{2}\angle(30\pi - 10\pi) = \mathbf{\frac{3}{2}\angle(20\pi)}$

iii) $z_1 z_2 z_3 = 3\angle(30\pi)\ 2\angle(10\pi)\ \angle(60\pi) = 6\angle(30\pi + 10\pi + 60\pi) = \mathbf{6\angle(100\pi)}$

iv) We have $z_3^2 = \left(\angle(60\pi)\right)^2 = \angle(120\pi)$ and so

$$\frac{z_2}{z_3^2} = \frac{2\angle(10\pi)}{\angle(120\pi)} = 2\angle(10\pi - 120\pi) = \mathbf{2\angle(-110\pi)}$$

v) $$\frac{z_2\, z_3}{z_1} = \frac{2\angle(10)\angle(60\pi)}{3\angle(30\pi)} = \frac{2}{3}\ \angle(10\pi + 60\pi - 30\pi)$$

$$= \mathbf{\frac{2}{3}\ \angle(40\pi)}$$

vi) $z_1^2 = \left(3\angle(30\pi)\right)^2 = 9\ \angle(60\pi)$ so

$$\frac{z_1^2\, z_3}{z_2} = \frac{9\angle(60\pi)\ \angle(60\pi)}{2\angle(10\pi)}$$

$$= \frac{9}{2}\ \angle(60\pi + 60\pi - 10\pi) = \mathbf{\frac{9}{2}\ \angle(110\pi)}$$